Some Positone Problems Suggested by Nonlinear Heat Generation

HERBERT B. KELLER* &
DONALD S. COHEN**

Communciated by JÜRGEN MOSER

1. Introduction. There is much current interest in boundary value problems containing positive linear differential operators and monotone functions of the dependent variable, see for example, M. A. Krasnosel'ski [1] and H. H. Schaefer [2]. We call such problems "positone" and shall examine here a particular class of them (which have been called non-linear eigenvalue problems in [2]).

One physical motivation for the problems we study concerns the temperature distribution in an object heated by the application of a uniform electric current, $i = \lambda^{1/2}$, (Joule heating). If the body is inhomogeneous with thermal conductivity $K(x)$, the electrical resistance, $R(x, T)$, of the object is a function of the temperature, $T(x, t)$, and if radiation is neglible, the resulting problem can be formulated, in some dimensionless form, as

$$\partial T/\partial t - \nabla \cdot (K(x)\nabla T) = \lambda R(x, T), \tag{1.1}$$

subject to appropriate initial and boundary conditions. In particular, we are interested in the steady states, their "stability", and their dependence upon the current, λ. This leads to problems of the form

$$-\nabla \cdot (K(x)\nabla T) = \lambda R(x, T), \tag{1.2}$$

subject to appropriate boundary conditions. In many cases of physical interest $R(x, T)$ is a monotone function of T (*i.e.*, the resistance increases with temperature) and only positive solutions are of interest. In some such cases it is known that a limiting current exists beyond which positive steady states do not exist. The value of this limiting current is of great interest.

* Research supported by the National Science Foundation under grant No. GP-4597 at the California Institute of Technology during the academic year 1965–66. Partly supported by the U. S. Army contract DA-31-124-ARO-D-360 with New York University.

** Research supported by the National Science Foundation under grant No. GP-4597 at the California Institute of Technology.

Journal of Mathematics and Mechanics, Vol. 16, No. 12 (1967).

The boundary conditions will always be assumed homogeneous. If, in the physical problem, they are not, say for example T is a prescribed function on the boundary, then we solve the steady state problem with zero current, $\lambda = 0$, to obtain the state $T_0(x)$. Subtract this from the desired state, $T(x)$, to obtain a problem with homogeneous boundary conditions for the difference, $u \equiv (T - T_0)$. This in effect changes the resistance term since

$$R(x, T) \equiv R(x, u + T_0) \equiv f(x, u),$$

say. It is also intuitively clear from these considerations that positive solutions, $u > 0$, are of interest (*i.e.*, the temperature increases when current is applied) and that the resistance should *not* vanish when $u \equiv 0$, *i.e.*, $f(x, 0) = R(x, T_0) > 0$.

For our study it is of no additional difficulty to treat more general equations of the form

$$Lu = \lambda f(x, u), \qquad x \in D, \tag{1.3}$$

where $x = (x_1, x_2, \cdots, x_m)$ and L is the uniformly elliptic, self-adjoint, second order operator

$$Lu \equiv -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u. \tag{1.4}$$

The coefficients $a_{ij}(x) = a_{ji}(x)$ are continuously differentiable, $a_0(x) \geqq 0$ is continuous and for all unit vectors $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$,

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geqq a > 0, \qquad x \in D. \tag{1.5}$$

The boundary conditions will be taken as

$$Bu \equiv \alpha(x)u(x) + \beta(x)\frac{\partial u(x)}{\partial \nu} = 0, \qquad x \in \partial D. \tag{1.6}$$

$$\alpha(x) \geqq 0, \not\equiv 0; \qquad \beta(x) \geqq 0.$$

Here $\partial/\partial\nu$ is the conormal derivative:

$$\frac{\partial u}{\partial \nu} \equiv \sum_{i,j=1}^{m} n_i(x)a_{ij}(x)\frac{\partial u}{\partial x_j}, \tag{1.7}$$

where $n(x) \equiv (n_1(x), \cdots, n_m(x))$ is the outer unit normal to ∂D at x. The functions $\alpha(x)$ and $\beta(x)$ are assumed piecewise continuous on ∂D; in fact, we require $\alpha(x) \equiv 1$, $\beta(x) \equiv 0$ on ∂D_1 where $\partial D_1 + \partial D_2 = \partial D$ and the measure of ∂D_1 is positive. The boundary is assumed so smooth that the strong maximum principle for L on D is valid, see [3].

In Section 2 we establish necessary and sufficient conditions for the existence of positive solutions to certain linear problems involving L and B. This yields the Positivity Lemma which is basic for all of our later results.

In Section 3 we investigate the nonlinear boundary value problem (1.3), (1.6) under rather mild monotonicity conditions on $f(x, u)$. (We will use the term "spectrum" in the sense of [1] and [2]. It is actually more analogous to part of the resolvent set. The point λ^* corresponds to the "bottom" of the spectrum in the usual sense.) The spectrum (*i.e.*, values of λ for which positive solutions exist) is completely characterized by means of an iteration procedure which yields the least or minimal positive solution when it converges. By means of a comparison theorem we then show that the spectrum is an interval, that the minimal solution is an increasing function of λ and obtain upper and lower bounds on the least upper bound, λ^*, of the spectrum.

The cases of concave and convex $f(x, u)$ are treated in Section 4 and more precise estimates of the spectrum are obtained. In particular for concave nonlinearities the spectrum is shown to be open and its upper limit, λ^*, is determined. Furthermore in this case the positive solutions are shown to be unique. These results are in marked contrast with special cases of convex $f(x, u)$ for which it is known that non-unique positive solutions exist and for which the spectrum is closed above. (We conjecture this to be true in general.)

Finally, in Section 5, we examine the stability of the positive solutions when considered as steady states of corresponding time dependent (parabolic) problems. We show that for $0 < \lambda < \lambda^*$ the minimal solutions are always stable and that for convex f they are more stable than any other positive solutions. Further as λ increases the relative stability of these minimal solutions increases if f is concave and decreases if f is convex.

2. A lemma on positive operators. It is easy to show that the operator L defined by (1.4)–(1.5) subject to homogeneous Dirichlet boundary conditions is positive. That is, if $\varphi(x)$ is twice continuously differentiable and satisfies $L\varphi(x) \geqq 0, \not\equiv 0$, in D and $\varphi(x) = 0$ on ∂D, then $\varphi(x) > 0$ in D. This result is a consequence of the maximum principle for elliptic operators [3]. However, we shall require a somewhat sharper and more general result which we state as the

Positivity Lemma. *Let $\rho(x)$ be positive and continuous on D and let $\varphi(x)$ be twice continuously differentiable and satisfy:*

$$(2.1) \qquad \begin{aligned} L\varphi - \lambda\rho(x)\varphi > 0, &\qquad x \in D, \\ B\varphi = 0, &\qquad x \in \partial D. \end{aligned}$$

Then $\varphi(x) > 0$ on D if and only if $\lambda < \mu_1$, where μ_1 is the principal (i.e., least) eigenvalue of

$$(2.2) \qquad \begin{aligned} L\psi - \mu\rho(x)\psi = 0, &\qquad x \in D, \\ B\psi = 0, &\qquad x \in \partial D. \end{aligned}$$

Proof. The sufficiency part of this lemma follows from the fact that the Green's function, G_λ, for $L - \lambda\rho(x)$ on D subject to $BG_\lambda = 0$ on ∂D is *positive* on D if $\lambda < \mu_1$. A proof of this fact follows from the work of Aronszajn and Smith [4] on reproducing kernels. However, by using a variational characterization of the solution of the appropriate boundary value problem, Bellman [5] has given a neat proof of the fact that G_λ is non-negative for a special ordinary differential operator. This idea can easily be extended and generalized to prove sufficiency in the present case, and we shall indicate it here.

Let us write (2.1) as

$$L\varphi - \lambda\rho(x)\varphi = p(x), \qquad x \in D, \tag{2.3}$$
$$B\varphi = 0, \qquad x \in \partial D,$$

where $p(x) > 0$ on D and is continuous there. The solution $\varphi(x)$ of this boundary value problem is also the function which minimizes the quadratic functional

$$I[\psi] \equiv Q[\psi] + \int_{\partial D_2} \frac{\alpha(x)}{\beta(x)} \psi^2(x)\, ds,$$

where

$$Q[\psi] \equiv \int_D \left\{ \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j} + [a_0(x) - \lambda\rho(x)]\psi^2 - 2p(x)\psi \right\} dx,$$

over the class of admissible functions $\mathfrak{A} \equiv \{$all piecewise continuously differentiable functions $\psi(x)$ on D which vanish on $\partial D_1\}$. By using the variational characterization of μ_1, we can easily show that if $\lambda < \mu_1$, the quadratic terms in $I[\psi]$ are positive definite for the class of admissible functions $\mathfrak{A}$. This fact insures the existence of a unique minimum which can then be shown to be a twice continuously differentiable solution of the boundary value problem (2.3). Conversely, any twice continuously differentiable solution of (2.3) is known to minimize $I[\psi]$ over $\mathfrak{A}$.

To show that the minimizing function, say $\varphi(x)$, is positive, suppose to the contrary that it is negative somewhere in D. Then, define an admissible function $\psi(x)$ by

$$\psi(x) \equiv |\varphi(x)|.$$

This does not affect the quadratic terms in $I[\psi]$, but it clearly diminishes the contribution from the term involving $p(x)$. This contradicts the fact that $\varphi(x)$ is the minimizing function, from which it follows that $\varphi(x) \geqq 0$ on D if $\lambda < \mu_1$. To show that $\varphi(x) > 0$ on D if $\lambda < \mu_1$, suppose that $\varphi(x) = 0$ at some point $x \in D$. Such a point would be a relative minimum at which $\partial\varphi(x)/\partial x_i = 0$, $i = 1, 2, \cdots, m$ and at which the matrix $(\partial^2\varphi(x)/\partial x_i\, \partial x_j)$ must be positive semi-definite. At this minimum the equation (2.3) reduces to

$$-\sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2\varphi(x)}{\partial x_i\, \partial x_j} = p(x) > 0,$$

which contradicts the fact that $(a_{ij}(x))$ is positive definite. Thus, $\varphi(x) > 0$ on D if $\lambda < \mu_1$.

To show necessity let $\varphi(x) > 0$ on D be a solution of (2.3) and let $\psi_1(x)$ be an eigenfunction of (2.2) corresponding to μ_1. It is well known that $\psi_1(x) \neq 0$ on D. Now, form the quantity $(\psi_1 L\varphi - \varphi L\psi_1)$, integrate it over D, and obtain, by partial integration and a use of the boundary value problems (2.2) and (2.3) satisfied by $\psi_1(x)$ and $\varphi(x)$, the relation

$$(\mu_1 - \lambda) \int_D \rho(x)\varphi(x)\psi_1(x)\, dx = \int_D p(x)\psi_1(x)\, dx.$$

Since both integrals are of the same sign, it follows that $\mu_1 > \lambda$, thus completing the proof. Q.E.D.

3. Existence and nonexistence of positive solutions. Under certain conditions on $f(x, u)$ we seek those values of λ for which the boundary value problem

$$\begin{aligned} Lu &= \lambda f(x, u), \qquad x \in D, \\ Bu &= 0, \qquad x \in \partial D, \end{aligned} \tag{3.1}$$

has positive solutions, $u(x) > 0$, $x \in D$. We call the set $\{\lambda\}$ of real values of λ for which positive solutions of (3.1) exist the spectrum of the problem (3.1), and the least upper bound of the spectrum is denoted by λ^*. The conditions to be imposed on f will be one or more of the following:

H–0: $f(x, \varphi)$ is continuous on the $m + 1$ dimensional half-cylinder $x \in D$, $\varphi \geqq 0$;

H–1: $f(x, 0) \equiv f_0(x) > 0$ on D;

H–2: $f(x, \varphi) > f(x, \psi)$ on D if $\varphi > \psi \geqq 0$.

The last condition is the first monotonicity requirement to be imposed on $f(x, u)$. Stronger restrictions will be imposed in Section 4.

We first observe that only positive λ are in the spectrum for a large class of nonlinearities including those satisfying H–0, 1, 2. More precisely, we have

Theorem 3.1. *Let $f(x, \varphi) > 0$ on D if $\varphi > 0$ and satisfy* H–0. *Then, only positive λ can be in the spectrum of* (3.1).

Proof. The proof is by contradiction. Assume that $u(x) > 0$ is a solution of (3.1) with $\lambda < 0$. Then, $\lambda f(x, u) < 0$ on D, and hence $-u(x)$ satisfies $L(-u) > 0$ on D, $B(-u) = 0$ on ∂D. From the Positivity Lemma (with $\lambda = 0$), we conclude that $-u(x) > 0$ on D which contradicts our assumption. The only solution of (3.1) with $\lambda = 0$ is $u(x) \equiv 0$, and the theorem follows. Q.E.D.

The existence of positive solutions for a large class of monotone nonlinearities is covered by

Theorem 3.2. *Let $f(x, \varphi)$ satisfy* H–0, 1, 2. *For any $\lambda > 0$ define the sequence $\{u_n(\lambda; x)\}$ by:*

$$u_0(x) \equiv 0,$$

$$\left.\begin{aligned} Lu_n(x) &= \lambda f(x, u_{n-1}(x)), \qquad x \in D, \\ Bu_n(x) &= 0, \qquad x \in \partial D. \end{aligned}\right\} \quad n = 1, 2, 3, \cdots \tag{3.2}$$

Then, $\lambda > 0$ *is in the spectrum of* (3.1) *if and only if the sequence* $\{u_n(\lambda; x)\}$ *is uniformly bounded. For* λ *in the spectrum this sequence converges uniformly and its limit, say*

$$\lim_{n\to\infty} [u_n(\lambda; x)] = \mathbf{u}(\lambda; x),$$

is the minimal positive solution of (3.1); *that is,* $\mathbf{u}(\lambda; x) > 0$ *and* $\mathbf{u}(\lambda; x) \leqq u(\lambda; x)$ *on* D *for any positive solution* $u(\lambda; x)$.

Proof. We first show, by induction, that the sequence defined in (3.2) is monotone increasing for $\lambda > 0$; that is,

$$u_{n+1}(\lambda; x) > u_n(\lambda; x), \qquad x \in D, \qquad n = 0, 1, 2, \cdots .$$

By condition H–1 we have, recalling that $u_0(x) \equiv 0$,

$$Lu_1(x) = \lambda f_0(x) > 0 \quad \text{in} \quad D,$$
$$Bu_1 = 0 \quad \text{on} \quad \partial D.$$

Hence by the Positivity Lemma it follows that $u_1(x) > 0$ on D. Assume the monotonicity established for all $n \leqq \nu$, say. Then, by condition H–2 and the inductive assumption we have

$$L[u_{\nu+1} - u_\nu] = \lambda[f(x, u_\nu) - f(x, u_{\nu-1})] > 0 \quad \text{on} \quad D,$$
$$B[u_{\nu+1} - u_\nu] = 0 \quad \text{on} \quad \partial D.$$

Now, the Positivity Lemma implies $u_{\nu+1} > u_\nu$ on D to conclude the induction.

If the iterates (3.2) are uniformly bounded then they converge, and their limit, say $\mathbf{u}(\lambda; x)$, is a positive function on D which is also uniformly bounded. That is $\mathbf{u}(\lambda; x) \leqq M$ on D for some positive number M. We now employ the Green's function, $G_0(x, \xi)$ for L on D subject to $BG_0 = 0$ for $x \in \partial D$ to write the iteration scheme (3.2) in the equivalent form: $u_0(x) \equiv 0$;

$$(3.3) \qquad u_n(\lambda; x) = \lambda \int_D G_0(x, \xi) f(\xi, u_{n-1}(\lambda; \xi))\, d\xi, \qquad n = 1, 2, 3, \cdots .$$

Now, $u_n(\lambda; x) \leqq \mathbf{u}(\lambda, x) \leqq M$ on D, $n = 1, 2, 3, \cdots$, and $G_0(x, \xi) f(\xi, u_{n-1}) < G_0(x, \xi) f(\xi, M)$ with

$$\int_D G_0(x, \xi) f(\xi, M)\, d\xi < \infty .$$

Thus, the Lebesgue bounded convergence theorem for Riemann integrals implies that the limit can be taken under the integral in (3.3) to conclude that

$$\mathbf{u}(\lambda; x) = \lambda \int_D G_0(x, \xi) f(\xi, \mathbf{u}(\lambda; \xi))\, d\xi.$$

It follows that $\mathbf{u}(\lambda; x)$ is a positive solution of (3.1), and so the sufficiency part of the theorem follows. Note that since $\mathbf{u}$ is continuous, the sequence $\{u_n(\lambda, x)\}$

is a sequence of continuous, monotone functions converging to a continuous limit. Thus, Dini's Theorem implies that the convergence is uniform.

Now, assume that $\lambda > 0$ is in the spectrum of (3.1) and let $u(\lambda; x)$ be some corresponding positive solution. Clearly, $u(\lambda; x) > u_0(\lambda; x) \equiv 0$ on D. Furthermore, if $u(\lambda; x) > u_{n-1}(\lambda; x)$ on D, then by H–2,

$$L[u - u_n] = \lambda[f(x, u) - f(x, u_{n-1})] > 0 \quad \text{on} \quad D,$$
$$B[u - u_n] = 0 \quad \text{on} \quad \partial D.$$

The Positivity Lemma and an induction now yield $u(\lambda; x) > u_n(\lambda; x)$ on D, $n = 0, 1, 2, \cdots$. Hence, the monotone sequence $\{u_n(\lambda; x)\}$ is uniformly bounded and as above converges uniformly to a positive solution, $\mathbf{u}(\lambda; x)$, which satisfies $u(\lambda; x) \geqq \mathbf{u}(\lambda; x)$ on D. We see that $\mathbf{u}(\lambda; x)$ is the minimal positive solution and the necessity part of the theorem has also been proved. Q.E.D.

The iteration scheme and characterization of the minimal solution in Theorem 3.2 lead to a variety of results on the existence and dependence of the minimal solution on λ. First, we have the basic comparison theorem,

Theorem 3.3. *Let $f(x, \psi)$ satisfy* H–0, 1, 2. *Let $F(x, \varphi)$ satisfy*

$$F(x, \varphi) > f(x, \psi) \quad \text{on} \quad D \quad \text{if} \quad \varphi > \psi \geqq 0;$$

and for some $\lambda > 0$ a positive solution $v(\lambda; x)$ exists for the problem

$$\begin{aligned} Lv &= \lambda F(x, v), \qquad x \in D, \\ Bv &= 0, \qquad x \in \partial D. \end{aligned} \tag{3.4}$$

Then, λ is in the spectrum of (3.1), *and the minimal positive solution of* (3.1) *satisfies*

$$\mathbf{u}(\lambda; x) \leqq v(\lambda; x) \quad \text{on} \quad D.$$

Proof. We prove, by induction, that the monotone increasing sequence of iterates in (3.2) satisfies $u_n(\lambda; x) < v(\lambda; x)$ on D. This follows as in the proof of Theorem 3.2 since $v(\lambda; x) > u_0(\lambda; x) \equiv 0$ and if $u_{\nu-1}(\lambda; x) < v(\lambda; x)$, then

$$L[v - u_\nu] = \lambda[F(x, v) - f(x, u_{\nu-1})] > 0 \quad \text{in} \quad D,$$
$$B[v - u_\nu] = 0 \quad on \quad \partial D.$$

Thus, $\lim_{n\to\infty} [u_n(\lambda; x)] = \mathbf{u}(\lambda; x)$ exists, is the minimal positive solution of (3.1), and satisfies $\mathbf{u}(\lambda; x) \leqq v(\lambda; x)$ in D. Q.E.D.

It is now a simple matter to show that the spectrum is an interval and that the minimal positive solution increases with λ. We state these results as the

Corollary 3.3.1. *Let $f(x, \varphi)$ satisfy* H–0, 1, 2 *and $\lambda' > 0$ be in the spectrum of* (3.1). *Then the interval $0 < \lambda \leqq \lambda'$ is in the spectrum and $\mathbf{u}(\lambda; x)$ is an increasing function of λ on the spectrum for each $x \in D$.*

Proof. For any fixed value of λ in the open interval $0 < \lambda < \lambda'$ we define

$$F(x, \varphi) \equiv \frac{\lambda'}{\lambda} f(x, \varphi).$$

Then for this value of λ the hypothesis of Theorem 3.3 is satisfied, say with $v(\lambda; x) \equiv \mathbf{u}(\lambda'; x)$. Hence λ is in the spectrum of (3.1). We also have by this theorem that

$$\mathbf{u}(\lambda; x) \leqq \mathbf{u}(\lambda'; x).$$

But then using H–2 it follows that

$$\begin{aligned} L[\mathbf{u}(\lambda'; x) - \mathbf{u}(\lambda; x)] &= \lambda' f(x, \mathbf{u}(\lambda'; x)) - \lambda f(x, \mathbf{u}(\lambda; x)) \\ &\geqq (\lambda' - \lambda) f(x, \mathbf{u}(\lambda; x)) \\ &> 0, \quad \text{on} \quad D. \end{aligned}$$

Of course $B[\mathbf{u}(\lambda'; x) - \mathbf{u}(\lambda; x)] = 0$ on ∂D and so by the Positivity Lemma: $\mathbf{u}(\lambda'; x) > \mathbf{u}(\lambda; x)$ on D. Clearly this holds for any two values $\lambda' > \lambda$ in the spectrum of (3.1). Q.E.D.

Some results concerning the extent of the spectrum also follow from the above theorem.

Corollary 3.3.2. Let $f(x, \varphi)$ satisfy H–0, 1, 2 *and in addition, for some positive function $F(x)$ on D,*

$$f(x, \varphi) < F(x) \quad \text{if} \quad \varphi > 0.$$

Then, all $\lambda > 0$ are in the spectrum of (3.1) *and thus $\lambda^* = \infty$.*

Proof. If we use the function $F(x)$ in place of $F(x, v)$ in problem (3.4), then we are assured that a positive solution exists for all $\lambda > 0$, namely

$$v(\lambda; x) = \lambda \int_D G_0(x, \xi) F(\xi) \, d\xi,$$

where $G_0(x, \xi)$ is the (positive) Green's function used in (3.3). The corollary now follows from Theorem 3.3 Q.E.D.

Corollary 3.3.3. Let $f(x, \varphi)$ satisfy H–0, 1, 2, *and for some positive functions $F(x)$ and $\rho(x)$:*

$$f(x, \varphi) < F(x) + \rho(x)\varphi \quad \text{on} \quad D \quad \text{for} \quad \varphi > 0.$$

Then, the spectrum of (3.1) *contains all λ in $0 < \lambda < \mu_1\{\rho\}$, where $\mu_1\{\rho\}$ is the principal eigenvalue of* (2.2). *The least upper bound λ^* on the spectrum is then bounded below by*

$$\mu_1\{\rho\} \leqq \lambda^*.$$

Proof. By the Positivity Lemma the problem (3.4) with the choice $F(x, v) \equiv F(x) + \rho(x)v$ has a positive solution, $v(\lambda; x)$, for each λ in $0 < \lambda < \mu_1\{\rho\}$. An application of Theorem 3.3 now yields the result. Q.E.D.

Note that Corollary 3.3.2 is a limiting form of Corollary 3.3.3 since, by the variational characterization of the principal eigenvalue of (2.2), we have in an obvious notation $\lim_{\rho\to 0}[\mu_1\{\rho\}] = \infty$.

A result on nonexistence is contained in

Corollary 3.3.4. *Let $f(x, \psi)$ and $F(x, \varphi)$ satisfy the hypothesis of Theorem* 3.3. *Let $\lambda^* < \infty$ be the least upper bound on the spectrum of* (3.1). *Then, the problem* (3.4) *has no positive solutions for $\lambda > \lambda^*$. In particular, if $f(x, \varphi)$ satisfies $f(x, \varphi) > F(x) + \rho(x)\varphi$ on D for $\varphi > 0$, with $F(x)$ and $\rho(x)$ positive, then $\lambda^* \leqq \mu_1\{\rho\}$, where $\mu_1\{\rho\}$ is the principal eigenvalue of* (2.2).

Proof. Assume that (3.4) has a positive solution for some $\lambda > \lambda^*$. Then, by Theorem 3.3 the problem (3.1) would have a positive solution for this value of λ. This contradicts the definition of λ^*, and the first part of the corollary follows.

To prove the second part of the corollary we note that the Positivity Lemma implies that the problem $L\varphi = \lambda[F(x) + \rho(x)\varphi]$ has no positive solutions for $\lambda > \mu_1\{\rho\}$. Now, an application of the first part of this corollary yields the result. Q.E.D.

Another nonexistence result which illustrates the importance of condition H–1 when $f(x, \varphi)$ is dominated by a linear function of φ is clearly shown in

Theorem 3.4. *Let $f(x, \varphi)$ satisfy* H–0, *and in addition, for some $\rho(x) > 0$ assume that $f(x, \varphi) < \rho(x)\varphi$ on D for $\varphi > 0$. Then, a positive solution of* (3.1) *does not exist for any λ in $0 < \lambda < \mu_1\{\rho\}$ where $\mu_1\{\rho\}$ is the principal eigenvalue of* (2.2).

Proof. Assume to the contrary that a positive solution, u, of (3.1) does exist for some λ in $0 < \lambda < \mu_1\{\rho\}$. For this solution we have $Bu = 0$ on ∂D and

$$Lu - \lambda\rho(x)u = \lambda[f(x, u) - \rho(x)u] < 0 \quad \text{on} \quad D.$$

An application of the Positivity Lemma to $(-u)$ yields that $(-u) > 0$ on D which contradicts the assumed positivity of $u(x)$. Q.E.D.

4. Concave and convex nonlinearities. In this section we shall require that $f(x, u)$ satisfy the strong monotonicity condition

$$\text{H} - 2'\text{:} \quad \frac{\partial f(x, \phi)}{\partial \phi} > 0 \quad \text{and continuous on } D \text{ for } \varphi > 0.$$

Clearly this implies condition H–2. Now, we have

Theorem 4.1. *Let $f(x, \varphi)$ satisfy* H–0, 1, 2′ *and be such that* (3.1) *has positive solutions for all λ in $0 < \lambda < \lambda^*$. Then each λ in this interval must satisfy $\lambda < \mu_1(\lambda)$ where $\mu_1(\lambda) \equiv \mu_1\{f_u(x, \mathbf{u}(\lambda; x))\}$ is the principal eigenvalue of*

$$(4.1) \qquad L\psi - \mu f_u(x, \mathbf{u}(\lambda; x))\psi = 0, \quad x \in D,$$
$$B\psi = 0, \quad x \in \partial D.$$

Proof. By employing the uniform convergence of the iterates in (3.2) to the minimal positive solution, it is not difficult to deduce that $\partial \mathbf{u}(\lambda; x)/\partial\lambda \equiv \mathbf{v}(\lambda; x)$ exists, is continuous in λ on $0 < \lambda < \lambda^*$, and hence satisfies the variational system

$$\begin{aligned} L\mathbf{v} - \lambda f_u(x, \mathbf{u})\mathbf{v} &= f(x, \mathbf{u}), \qquad x \in D, \\ B\mathbf{v} &= 0, \qquad x \in \partial D. \end{aligned} \tag{4.2}$$

From Corollary 3.3.1 we know that $\mathbf{u}(\lambda; x)$ is an increasing function of λ for $x \in D$ which implies that $\mathbf{v}(\lambda; x) \geqq 0$ on D. To show that $\mathbf{v}(\lambda; x)$ is strictly positive on D, we note that since $\mathbf{u}(\lambda; x)$ is positive on D, it follows from conditions H–1, 2 that $f(x, \mathbf{u}(\lambda; x)) > 0$ on D. If $\mathbf{v}(\lambda; x) = 0$ at some point $x \in D$, such a point must be a relative minimum at which $\partial\mathbf{v}/\partial x_i = 0$, $i = 1, 2, \cdots, m$, and at which the matrix $(\partial^2\mathbf{v}/\partial x_i\, \partial x_j)$ must be positive semi-definite. At this point the equation (4.2) reduces to

$$-\sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 \mathbf{v}}{\partial x_i\, \partial x_j} = f(x, \mathbf{u}) > 0,$$

which contradicts the fact that $(a_{ij}(x))$ is positive definite. Thus, $\mathbf{v}(\lambda; x) > 0$ on D.

We have previously observed that $f(x, \mathbf{u}) > 0$ on D, and similarly from the positivity of $\mathbf{u}(\lambda; x)$ we note, by H–2′, that $f_u(x, \mathbf{u}) > 0$ on D. The Positivity Lemma can now be applied to the problem (4.2) satisfied by $\mathbf{v} > 0$ to conclude that $\lambda < \mu_1(\lambda)$. Q.E.D.

With the notation $\mathbf{u}(0; x) \equiv 0$ as the minimal non-negative solution of (3.1) for $\lambda = 0$ we may use (4.1) to define the eigenvalue $\mu_1(0) \equiv \mu_1\{f_u(x, 0)\}$, even though $\lambda = 0$ cannot be in the spectrum.

Theorem 4.1 required only the strong monotonicity of $f(x, u)$. If in addition the nonlinearity is concave or convex the function $\mu_1(\lambda)$ can be studied in more detail. We say that $f(x, u)$ is (a) *concave* or (b) *convex*, respectively, if it satisfies H–2′ and

H–3a: $f_u(x, \varphi) < f_u(x, \psi)$ on D if $\varphi > \psi \geqq 0$ (concave),

or

H–3b: $f_u(x, \varphi) > f_u(x, \psi)$ on D if $\varphi > \psi \geqq 0$ (convex).

If $f(x, u)$ satisfies H–0, 1 and is concave, then clearly

$$f(x, \varphi) < f_0(x) + f_u(x, 0)\varphi, \qquad \varphi > 0, \tag{4.3a}$$

or if it satisfies H–0, 1 and is convex, then

$$f(x, \varphi) > f_0(x) + f_u(x, 0)\varphi, \qquad \varphi > 0. \tag{4.3b}$$

An application of Corollary 3.3.3 to the case of concave f reveals that the spectrum of (3.1) contains the interval $0 < \lambda < \mu_1(0)$ and that $\lambda^* \geqq \mu_1(0)$. On the other hand, for f convex the Corollary 3.3.4 and the bound (4.3b) imply that the least upper bound on the spectrum of (3.1) is bounded above by $\lambda^* \leqq \mu_1(0)$. To improve these bounds we note the result in

Corollary 4.1.1. *Let $f(x, \varphi)$ satisfy* H–0, 1, 2′, 3a (*or* 3b). *If* (3.1) *has the spectrum* $0 < \lambda < \lambda^*$, *then $\mu_1(\lambda)$ is an increasing* (*or decreasing*) *function of λ on this interval. Furthermore, for f concave $\mu_1(\lambda) < \lambda^*$, and for f convex $\mu_1(\lambda) > \lambda^*$, on* $0 < \lambda < \lambda^*$.

Proof. From the variational characterization of the principal eigenvalue, $\mu_1(\lambda)$, of the problem (4.1) we can write

$$\mu_1(\lambda) = \min_{\psi(x)\,\varepsilon\,\mathfrak{M}} \left[\frac{(\psi, L\psi)}{(\psi, f_u(x, \mathbf{u}(\lambda; x))\psi)} \right], \tag{4.4}$$

where the obvious inner product used is

$$(\psi, \varphi) = \int_D \psi(x)\varphi(x)\, dx,$$

and the class $\mathfrak{M}$ of admissible functions can be taken as

$$\mathfrak{M} \equiv \{\psi(x) \mid \psi(x) > 0 \text{ on } D,\ \psi(x)\ \varepsilon\ C(\bar{D}) \cap C'(D),\ \psi(x) = 0 \text{ on } \partial D_1\}.$$

Here we recall that ∂D_1 is that portion of ∂D on which $\beta(x) \equiv 0$ and $\alpha(x) \equiv 1$.

From Corollary 3.3.1 we know that the minimal positive solutions are increasing functions of λ on $0 < \lambda < \lambda^*$. Thus, if $f(x, u)$ is concave and if $0 < \lambda < \lambda' < \lambda^*$ we must have

$$f_u(x, \mathbf{u}(\lambda; x)) > f_u(x, \mathbf{u}(\lambda'; x)) \quad \text{on} \quad D.$$

Moreover, since $\psi\ \varepsilon\ \mathfrak{M}$ implies $\psi(x) > 0$ on D, it follows from the above and (4.4) that $\mu_1(\lambda) < \mu_1(\lambda')$. The opposite inequality for convex f follows in an obvious manner.

We have observed, just before the statement of this corollary, that $\mu_1(0) \geqq \lambda^*$ in the case of convex f. But by Theorem 4.1 it follows that $\lambda < \mu_1(\lambda)$ in $0 < \lambda < \lambda^*$. Thus, since $\mu_1(\lambda)$ is a decreasing function of λ, we may conclude that $\mu_1(\lambda) > \lambda^*$ on $0 \leqq \lambda < \lambda^*$ for convex f.

If f is concave, then $\mu_1(0) \leqq \lambda^*$ as previously indicated. However, in this case, for any λ in $0 < \lambda < \lambda^*$ and any $\varphi(x) > 0$ on D we must have

$$\begin{aligned} f(x, \varphi) &\leqq f(x, \mathbf{u}) + (\varphi - \mathbf{u})f_u(x, \mathbf{u}) \\ &< f_0(x) + [f_u(x, 0) - f_u(x, \mathbf{u})]\mathbf{u}(\lambda; x) + f_u(x, \mathbf{u})\varphi. \end{aligned} \tag{4.5}$$

Now, we apply Corollary 3.3.3 to conclude that $\mu_1(\lambda) \leqq \lambda^*$ on $0 < \lambda < \lambda^*$ for concave f. Q.E.D.

The function $\mu_1(\lambda)$ has been defined only for $0 \leqq \lambda < \lambda^*$. However, under the hypotheses of Theorem 4.1 we may define a function $\mu_1(\lambda)$ for all $\lambda \geqq 0$ which agrees with the previous definition when $\lambda < \lambda^*$. For this purpose let $\{u_n(\lambda; x)\}$ be the sequence defined in (3.2) for any $\lambda \geqq 0$. Then, define $\mu_{1,n}(\lambda)$ to be the principal eigenvalue of the problem

$$\begin{aligned} L\psi - \mu f_u(x, u_n(\lambda; x))\psi &= 0, \qquad x\ \varepsilon\ D, \\ B\psi &= 0, \qquad x\ \varepsilon\ \partial D. \end{aligned} \tag{4.6a}$$

Finally, we define $\mu_1(\lambda)$ as

(4.6b) $$\mu_1(\lambda) = \lim_{n\to\infty} [\mu_{1,n}(\lambda)].$$

To show that (4.6) defines $\mu_1(\lambda)$ for all λ we write the equivalent variational characterization

$$\mu_1(\lambda) = \lim_{n\to\infty} \min_{\psi\in\mathfrak{M}} \left[\frac{(\psi, L\psi)}{(\psi, f_u(x, u_n(\lambda; x))\psi)}\right].$$

Clearly, if $\lambda < \lambda^*$, the least upper bound of the spectrum, then by Theorem 3.2 the sequence $\{u_n(\lambda; x)\}$ converges to $\mathbf{u}(\lambda; x)$ and we obtain the same values defined in Theorem 4.1. If $\lambda > \lambda^*$, the sequence does not converge, but since it is monotone increasing and the $u_n(\lambda; x)$ are uniformly continuous on $\bar{D}$, we must have $\lim_{n\to\infty} [u_n(\lambda; x)] = \infty$ on D. Now either $\lim_{M\to\infty} [f_u(x, M)] = \infty$ or $r(x) \geqq 0$. In the former case $\mu_1(\lambda) = 0$ if $\lambda > \lambda^*$, and in the latter case $\mu_1(\lambda) = m$ if $\lambda > \lambda^*$, where m is the principal eigenvalue of $L\psi - \mu r(x)\psi = 0$ on D, $B\psi = 0$ on ∂D. Thus, in any event, $\mu_1(\lambda)$ is constant for all $\lambda > \lambda^*$. The behavior of this extended function $\mu_1(\lambda)$ is sketched in figure 1 for both concave and convex f. (See however Corollary 4.1.2, Theorem 4.2 and the remark following.)

Some important facts concerning the limit of the spectrum for the case of concave f are contained in

Corollary 4.1.2. *Let $f(x, \varphi)$ satisfy* H–0, 1, 2′ *and* 3a (*i.e. f is concave*). *If λ^* is the least upper bound on the spectrum of* (3.1) *then*

(4.7) $$\lim_{\lambda\uparrow\lambda^*} \mu_1(\lambda) = \lambda^*,$$

and λ^* is not a point of the spectrum (*i.e. the spectrum is open*).

Proof. It is clear from the continuous (in fact differentiable) dependence of $\mathbf{u}(\lambda; x)$ on λ and, say, the characterization (4.4) that $\mu_1(\lambda)$ is a continuous function of λ on $0 < \lambda < \lambda^*$. From Theorem 4.1 and Corollary 4.1.1 we have for concave f and $\lambda < \lambda^*$ that:

$$\lambda < \mu_1(\lambda) < \lambda^*.$$

Now (4.7) follows on letting λ approach λ^* from below.

Suppose that λ^* is in the spectrum of (3.1). Then a corresponding minimal positive solution $\mathbf{u}(\lambda^*, x)$ exists and is finite. By (4.7) and the continuous dependence on λ we have that

$$\lambda^* = \mu_1\{f_u(x; \mathbf{u}(\lambda^*, x)\}.$$

Now since $f(x, \varphi)$ is concave it follows, just as in the derivation of (4.5), that

$$f(x, \varphi) < F(x) + \rho(x)\varphi \quad \text{on} \quad D \quad \text{for} \quad \varphi > 0,$$

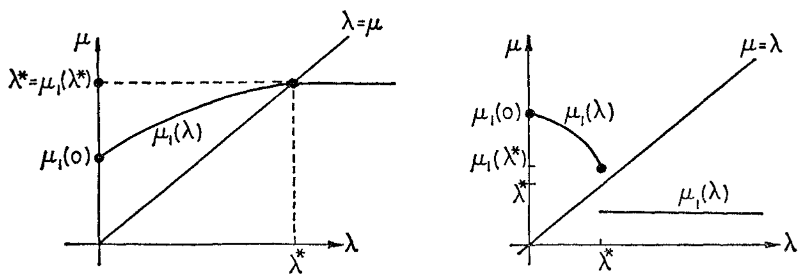
where for any $\psi(x) > 0$ on D:

$$F(x) \equiv f_0(x) + [f_u(x, 0) - f_u(x, \psi(x))]\psi(x),$$
$$\rho(x) \equiv f_u(x, \psi(x)).$$

Pick some smooth bounded function $\psi(x) > \mathbf{u}(\lambda^*, x)$ on D and now apply Corollary 3.3.3 to conclude that $\lambda^* \geqq \mu_1\{f_u(x, \psi(x))\}$. However this is a contradiction since $f_u(x; \mathbf{u}(\lambda^*, x)) > f_u(x, \psi(x))$ on D and the variational characterization of the principal eigenvalue μ_1 then implies

$$\mu_1\{f_u(x, \mathbf{u}(\lambda^*, x))\} < \mu_1\{f_u(x, \psi(x))\}. \qquad \text{Q.E.D.}$$

The above proof suggests that the minimal positive solutions become unbounded for concave f as $\lambda \to \lambda^*$. Then the extended function $\mu_1(\lambda)$ would be continuous as is conjectured and indicated in figure 1a. However in cases of



(a) Concave f (b) Convex f

Figure I

convex f which have been solved explicitly [6], the solutions remain bounded as $\lambda \to \lambda^*$ while $\partial u/\partial\lambda \to \infty$. In these cases λ^* is a point of the spectrum. There is reason to conjecture that 4.7 also holds for convex nonlinearities.

Another fundamental difference between concave and convex nonlinearities concerns the uniqueness of positive solutions. For the concave case we have uniqueness as in

Corollary 4.1.3. Let $f(x, \varphi)$ satisfy H–0, 1, 2′ *and* 3a *(i.e. f is concave). Then positive solutions of* (3.1) *are unique for each λ in* $0 < \lambda < \lambda^*$.

Proof. Let $\mathbf{u}(\lambda; x)$ be the minimal positive solution of (3.1) for some fixed λ in $0 < \lambda < \lambda^*$. If some other positive solution $u(\lambda; x)$ exists for this value of λ it must satisfy

$$u(\lambda; x) - \mathbf{u}(\lambda; x) \geqq 0, \not\equiv 0 \quad \text{on} \quad D.$$

Furthermore $B[u - \mathbf{u}] = 0$ on ∂D and from the concavity of f we deduce that, in D,

$$\begin{aligned} L[u - \mathbf{u}] &= \lambda[f(x, u) - f(x, \mathbf{u})] \\ &= \lambda f_u(x, \mathbf{u} + \theta[u - \mathbf{u}])[u - \mathbf{u}], \qquad 0 < \theta(x) < 1, \\ &\leqq \lambda f_u(x, \mathbf{u})[u - \mathbf{u}]. \end{aligned}$$

From Theorem 4.1 it follows that $\lambda < \mu_1\{f_u(x, \mathbf{u})\}$ and so we may apply an obvious slightly weakened form of the Positivity Lemma to conclude that $u(\lambda; x) - \mathbf{u}(\lambda; x) \leqq 0$. It therefore follows that $u - \mathbf{u} \equiv 0$. Q.E.D.

In some special cases of convex nonlinearities [6] the positive solutions are known to be nonunique.

Finally we may actually determine λ^* for many concave nonlinearities by means of

Theorem 4.2. *Let* $f(x, \varphi)$ *satisfy* H-0, 1, 2′, 3a *and in addition:*

$$\lim_{\varphi\to\infty} f_\varphi(x; \varphi) = \rho(x) \quad on \quad D.$$

Then the limit of the spectrum of (3.1) *is* $\lambda^* = \mu_1\{\rho\}$ *where we understand that* $\mu_1\{\rho\} = \infty$ *if* $\rho(x) \equiv 0$.

Proof. By the concavity of f and the limit condition it easily follows that there exist positive functions $F_1(x)$ and $F_2(x)$ such that, on D,

$$F_1(x) + \rho(x)\varphi < f(x, \varphi) < F_2(x) + \rho(x)\varphi,$$

for all $\varphi > 0$. The upper bound is determined as in (4.5) and we may take $F_1(x) \equiv f(x, 0)$. Corollaries 3.3.3 and 3.3.4 immediately imply that $\lambda^* \geqq \mu_1\{\rho\}$ and $\lambda^* \leqq \mu_1\{\rho\}$, respectively, so that $\lambda^* = \mu_1\{\rho\}$. In addition it follows from Corollary 3.3.2 that $\lambda^* = \infty$ if $\rho(x) \equiv 0$. Q.E.D

It should be observed that if, as we have previously conjectured, $\lim_{\lambda\uparrow\lambda^*} \mathbf{u}(\lambda; x) = \infty$ for concave f then the limit in (4.7) gives the same value for λ^* as that in Theorem 4.2.

5. Stability of positive solutions. Any solution $u(\lambda; x)$ of the boundary value problem (3.1) may be regarded as a steady state solution of the mildly non-linear parabolic problem.

$$\begin{aligned} &\frac{\partial U}{\partial t} + LU = \lambda f(x, U), && x \in D, \quad t > 0, \\ &BU = 0, && x \in \partial D, \quad t > 0, \\ &U(x, 0) = U_0(x), && x \in D. \end{aligned} \tag{5.1}$$

We shall give a more precise definition later, but roughly, we say that $u(\lambda; x)$ is stable if for all initial data of the form

$$U_0(x) = u(\lambda; x) + \epsilon V(x), \tag{5.2}$$

the solution of (5.1) decays exponentially in t to $u(\lambda; x)$ to first order in ϵ.

Assuming a solution of (5.1), (5.2) of the form

$$U(x, t) = u(\lambda; x) + \epsilon v(x)e^{-\alpha t} + O(\epsilon^2),$$

we find, to first order in ϵ, that α and $v(x)$ must satisfy:

$$\begin{aligned} &Lv - [\alpha + \lambda f_u(x, u)]v = 0, && x \in D, \\ &Bv = 0, && x \in \partial D. \end{aligned} \tag{5.3}$$

Clearly, non-trivial solutions $v \not\equiv 0$ exist if and only if α is an eigenvalue of (5.3) and $v = v(\alpha; x)$ is the corresponding eigenfunction. If the eigenfunctions of (5.3) are complete in some sense, then for some coefficients a_n,

$$V(x) = \sum_n a_n v(\alpha_n ; x)$$

and the solution, to first order in ϵ, of (5.1), (5.2) is

$$U(x, t) = u(\lambda; x) + \epsilon \sum_n a_n v(\alpha_n ; x) e^{-\alpha_n t} + O(\epsilon^2).$$

Thus, we are motivated to adopt the following

Definitions. A solution $u(\lambda; x)$ of (3.1) is *stable* if the principal eigenvalue $\alpha \equiv \alpha_1$ of (5.3) is positive; it is *unstable* if α_1 is negative; and it is *neutrally stable* if $\alpha_1 = 0$. For any set of solutions of (3.1) that solution with the largest principal eigenvalue in (5.3) is *relatively more stable.*

It should be noted that we have assumed $f(x, u)$ to be continuously differentiable with respect to u. We maintain this assumption throughout this section.

Theorem 5.1. *Let $f(x, \varphi)$ satisfy* H–0, 1, 2′ *and be such that* (3.1) *has the spectrum* $0 < \lambda < \lambda^*$ *or* $0 < \lambda \leqq \lambda^*$. *Then, the minimal positive solutions of* (3.1) *are stable in* $0 < \lambda < \lambda^*$. *If, in addition, $f(x, u)$ is convex, the minimal positive solution is relatively more stable than any other positive solutions with the same values of λ. Furthermore, if $f(x, u)$ is concave* (*or convex*), *the relative stability of the set of minimal positive solutions on* $0 < \lambda < \lambda^*$ *increases* (*or decreases*) *as λ increases.*

Proof. Let $u(\lambda; x)$ be some positive solution of (3.1), and denote the corresponding principal eigenvalue of (5.3) by $\alpha(\lambda)$. Then, with the usual inner product notation the variational characterization of the principal eigenvalue yields

$$\alpha(\lambda) = \min_{\varphi(x) \in \mathfrak{M}} \left[\frac{(\varphi, L\varphi) - \lambda(\varphi, f_u(x, u(\lambda; x))\varphi)}{(\varphi, \varphi)} \right]. \tag{5.4}$$

Now, recalling (4.4), we have for any $\varphi(x) \in \mathfrak{M}$ and any λ in $0 < \lambda < \lambda^*$,

$$(\varphi, L\varphi) \geqq \mu_1(\lambda)(\varphi, f_u(x, \mathbf{u}(\lambda; x))\varphi). \tag{5.5}$$

Thus, if $\boldsymbol{\alpha}(\lambda)$ is the eigenvalue corresponding to the minimal positive solution, $\mathbf{u}(\lambda; x)$, we obtain from (5.4) and (5.5) that

$$\boldsymbol{\alpha}(\lambda) \geqq \min \left[(\mu_1(\lambda) - \lambda) \frac{(\varphi, f_u(x, \mathbf{u}(\lambda; x))\varphi)}{(\varphi, \varphi)} \right].$$

By Theorem 4.1 we have $\mu_1(\lambda) - \lambda > 0$ and by H–2′ and $\mathbf{u} > 0$ on D we have $f_u(x, \mathbf{u}) > 0$ on D. It follows, since $\varphi(x) > 0$ on D, that $\boldsymbol{\alpha}(\lambda) > 0$, and thus, the minimal positive solutions are stable in $0 < \lambda < \lambda^*$.

Now, let $f(x, u)$ be convex. Then, since any positive solution, $u(\lambda; x)$ which is *not* minimal satisfies $u(\lambda; x) \geqq \mathbf{u}(\lambda; x)$ on D, we must have $f_u(x, \mathbf{u}) \leqq f_u(x, u)$ on D. The variational characterization (5.4) then implies that $\boldsymbol{\alpha}(\lambda) \geqq \alpha(\lambda)$ which establishes that the minimal positive solution is relatively more stable than any other positive solution.

Finally, Corollary 3.3.1 asserts that $\mathbf{u}(\lambda; x)$ increases as λ increases for each $x \in D$. Thus, if f is convex, $f_u(x, \mathbf{u})$ increases as λ increases. The variational principle (4.4) then implies that $\boldsymbol{\alpha}(\lambda)$ decreases as $f_u(x, \mathbf{u})$ increases, thus concluding the proof of the fact that if f is convex, the stability of the minimal positive solution decreases as λ increases. The opposite result obviously holds for f concave. Q.E.D.

Clearly, we have the following interpretation of the above theorem in terms of the nonlinear heat conduction problems mentioned in the introduction: In the cases where the steady state solutions are not unique, namely for convex f, the "cooler" positive steady states are more stable. Perhaps a somewhat surprising result is that for concave f, although the stability of the *unique* positive solution on $0 < \lambda < \lambda^*$ increases as the current, λ, increases, no positive solution exists when $\lambda > \lambda^*$. If, as we conjecture for convex f, the limit $\lambda^* = \mu_1(\lambda^*)$ then it is not difficult to show that $\boldsymbol{\alpha}(\lambda^*) = 0$, so that the limiting minimal positive solution has neutral stability. This has been demonstrated for a special case in [6].

References

[1] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd., Groningen, The Netherlands, 1964.

[2] H. H. Schaefer, *Some nonlinear eigenvalue problems*, Symposium on Nonlinear Problems, edited by R. E. Langer, University of Wisconsin Press, 1963.

[3] R. Courant & D. Hilbert, *Methods of Mathematical Physics*, vol. II, Interscience, New York, 1962.

[4] N. Aronazajn & K. Smith, Characterization of positive reproducing kernals. Applications to Green's functions, *Amer. J. Math.*, 79 (1957) 611–622.

[5] R. Bellmann, On the non-negativity of Green's functions, *Boll. d'Unione Mate.*, **12** (1957) 411–413.

[6] D. D. Joseph, Non-linear heat generation and stability of the temperature distribution in conduction solids, *Int. J. Heat Mass Transfer*, 8 (1965) 281–288.

Courant Institute of Mathematical Sciences
New York University
and
California Institute of Technology

Date Communicated: October 14, 1966